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Generalized Two-Mode Squeezed States and Some Nonclassical Characteristics

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Abstract A class of generalized two-mode squeezed states $|\phi\rangle$ is presented, which are generated from the generalized two-mode squeezing operator $U(\gamma, \lambda)$ acting on the two-mode coherent state $|\alpha_1, \alpha_2\rangle$. We first investigate some mathematical properties of $U(\gamma, \lambda)$ including the squeezing transformation under $U(\gamma, \lambda)$, ket-bra integral form in the coordinate representation, normally ordered form. Then we evaluate some nonclassical characteristics of the state $|\phi\rangle$ such as higher-order squeezing behavior, entanglement analysis and analytical expression of the Wigner function.

Keywords Generalized two-mode squeezed state · Higher-order squeezing · Entanglement · Wigner function

1 Introduction

In the last years, considerable attention in quantum optics has been paid to a class of optical field states that are called squeezed states [1], where the fluctuation in one quadrature is less than that of the vacuum state [2]. Efforts to generate squeezed states of light are motivated by the potential applications of squeezed light in optical communication and ultra-sensitive detection systems, which are presently quantum noise limited [3]. In principle several schemes for producing squeezed states have been analyzed, such as parametric amplifiers [4], four-wave mixing [5], resonance fluorescence [6], etc. It is well known that the two-mode squeezed state [7, 8], such as two-mode squeezed vacuum state, is a highly correlated two-mode state, which has been applied in quantum teleportation [9, 10] and quantum dense coding [11]. A state generated by applying the squeezing operator to the

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two-mode vacuum first, followed by the displacement operator, is called a two-mode coherent squeezed state. This is Stoler's way of generating squeezed states [12, 13]. On the other hand, if the order of the two operators is reversed, a two-mode squeezed coherent state is obtained, which is named Yuen's way of generating squeezed states [14].

Recently, many groups have devoted to the research on nonclassical properties of two-mode squeezed states [15–23]. Lee [15] and Caves' group [16] have studied the nonclassical photon statistics of two-mode squeezed states, respectively. The phase properties of correlated two-mode squeezed states are also discussed in [16] and [17]. Ping *et al.* [22] have investigated the entanglement properties of two-mode squeezed coherent states. In addition, some generalized two-mode squeezed states [24–27] have been constructed, which are great progress in quantum optics. Fan has proposed a generalized two-mode squeezed state, called one- and two-mode combination squeezed state [24], whose statistical properties are recently studied in [28]. Another generalized two-mode squeezed states are presented by Abdalla [25] and Deng's group [26], respectively.

In the present paper, we concentrate our attention on the generalized two-mode squeezed state denoted by $|\phi\rangle$, which are generated from the generalized two-mode squeezing operator $U(\gamma,\lambda)$ [27],

$$U(\gamma,\lambda)=S_1(\gamma)S_2(\gamma)S(\lambda), \tag{1}$$

where $S_j(\gamma)$ $(j=1,2)$ with the squeezing parameter γ is the single-mode squeezing operator for the jth mode,

$$S_j(\gamma)=\exp\left[\frac{1}{2}\gamma(a_j^2-a_j^{\dagger 2})\right], \tag{2}$$

and $S(\lambda)$ with the squeezing parameter λ are the well-known two-mode squeezing operator,

$$S(\lambda)=\exp\left[\lambda\left(a_1a_2-a_1^\dagger a_2^\dagger\right)\right]. \tag{3}$$

Our main task is to investigate some mathematical properties of the operator $U(\gamma,\lambda)$ in (1) and evaluate some nonclassical characteristics of the state $|\phi\rangle=U(\gamma,\lambda)|\alpha_1,\alpha_2\rangle$, where $|\alpha_1,\alpha_2\rangle$ is two-mode coherent state.

This paper is organized as follows. In Sect. 2, we first give the corresponding squeezing transformation under the operator $U(\gamma,\lambda)$. With the help of the technique of integration with an order product (IWOP), we found that $U(\gamma,\lambda)$ is a ket-bra integral form in the coordinate representation itself, which is a physically appealing form. Its normally ordered form and the corresponding squeezed state are obtained as well. In Sect. 3, we turn our attention to evaluate the higher-order squeezing behavior for the generalized two-mode squeezed state $|\phi\rangle$. It is found that the state $|\phi\rangle$ can exhibit a stronger form of higher-order squeezing than the usual two-mode squeezed state. In Sect. 4, entanglement for the state $|\phi\rangle$ is studied using the total variance of a pair of Einstein-Podolsky-Rosen (EPR)-like operators. We devote Sect. 5 to deriving the analytical expression of the Wigner function for the state $|\phi\rangle$ by using the Weyl ordering invariance under similar transformation. This approach seems very simply. Finally, in Sect. 6 we give our conclusions.

2 Mathematical Properties of $U(\gamma,\lambda)$

In this section, we give the corresponding squeezing transformation under the operator $U(\gamma,\lambda)$. By virtue of the technique of IWOP [29, 30], we obtain the normally ordered form of $U(\gamma, \lambda)$ and get the corresponding squeezed state. Note that in the following calculations we shall assume that γ and λ are real squeezing parameter.

To begin with, we derive the squeezing transformation under the operator $U(\gamma, \lambda)$. Using the squeezing transformation of the single-mode and two-mode squeezing operators, respectively,

$$S_j^{-1}(\gamma) a_j S_j(\gamma) = a_j \cosh\gamma - a_j^{\dagger} \sinh\gamma, \quad j = 1, 2, \tag{4}$$

and

$$S^{-1}(\lambda) a_1 S(\lambda) = a_1 \cosh\lambda - a_2^{\dagger} \sinh\lambda, \qquad S^{-1}(\lambda) a_2 S(\lambda) = a_2 \cosh\lambda - a_1^{\dagger} \sinh\lambda, \tag{5}$$

we easily obtain the following combination squeezing transformation under $U(\gamma, \lambda)$

$$U^{-1} a_1 U = \cosh\gamma \left(a_1 \cosh\lambda - a_2^{\dagger} \sinh\lambda\right) - \sinh\gamma \left(a_1^{\dagger} \cosh\lambda - a_2 \sinh\lambda\right), \tag{6}$$

and

$$U^{-1} a_2 U = \cosh\gamma \left(a_2 \cosh\lambda - a_1^{\dagger} \sinh\lambda\right) - \sinh\gamma \left(a_2^{\dagger} \cosh\lambda - a_1 \sinh\lambda\right). \tag{7}$$

Further, according to the coordinate operator and the momentum operator, respectively,

$$Q_j = \frac{1}{\sqrt{2}} \left(a_j + a_j^{\dagger}\right), \qquad P_j = \frac{1}{i\sqrt{2}} \left(a_j - a_j^{\dagger}\right), \tag{8}$$

it is readily obtained that

$$U^{-1} Q_1 U = e^{-\gamma} (Q_1 \cosh\lambda - Q_2 \sinh\lambda), \tag{9}$$

$$U^{-1} Q_2 U = e^{-\gamma} (Q_2 \cosh\lambda - Q_1 \sinh\lambda), \tag{10}$$

$$U^{-1} P_1 U = e^{\gamma} (P_1 \cosh\lambda + P_2 \sinh\lambda), \tag{11}$$

and

$$U^{-1} P_2 U = e^{\gamma} (P_2 \cosh\lambda + P_1 \sinh\lambda). \tag{12}$$

In order to construct the generalized two-mode squeezed state, we need the normally ordered form of $U(\gamma, \lambda)$. For this purpose, recall that the ket-bra integral form of $S_j(\gamma)$ and $S(\lambda)$ are obtained in the coordinate representation [31],

$$S_j(\gamma) = e^{-\frac{\gamma}{2}} \int dq_j |\exp(-\gamma) q_j\rangle\langle q_j| \tag{13}$$

and

$$S(\lambda) = \int dq_1 dq_2 |q_1 \cosh\lambda - q_2 \sinh\lambda, q_2 \cosh\lambda - q_1 \sinh\lambda\rangle\langle q_1, q_2|, \tag{14}$$

respectively, where $|q_1, q_2\rangle \equiv |q_1\rangle \otimes |q_2\rangle$, $|q_j\rangle$ $(j = 1, 2)$ is the eigenstate of coordinate operator Q_j, which in Fock space is expressed by

$$|q_j\rangle = \pi^{-1/4} \exp\left[-\frac{q_j^2}{2} + \sqrt{2} q_j a_j^{\dagger} - \frac{a_j^{\dagger 2}}{2}\right] |0\rangle. \tag{15}$$

Due to the orthogonality of coordinate eigenstate, i.e., $\langle q_1', q_2'|q_1, q_2\rangle = \delta(q_1 - q_1')\delta(q_2 - q_2')$, substituting (13) and (14) into (1), we have

$$U(\gamma,\lambda) = e^{-\gamma}\int dq_1dq_2|e^{-\gamma}(q_1\cosh\lambda - q_2\sinh\lambda), e^{-\gamma}(q_2\cosh\lambda - q_1\sinh\lambda)\rangle\langle q_1, q_2| \\ = e^{-\gamma}\int dq_1dq_2\left|R\begin{pmatrix} q_1 \\ q_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2\end{pmatrix}\right|, \tag{16}$$

where

$$R = \begin{pmatrix} e^{-\gamma}\cosh\lambda & -e^{-\gamma}\sinh\lambda \\ -e^{-\gamma}\sinh\lambda & e^{-\gamma}\cosh\lambda \end{pmatrix}, \quad \det R = 1. \tag{17}$$

It is clear from (16) that $U(\gamma,\lambda)$ is the projection operators in an integral form, which is different from (2.7) in [24]. When $e^{-\gamma} = 1$, $\sinh\gamma = 0$, (6) and (7) reduce to (5), while (16) reduces to (14). Considering the unitarity of $U(\gamma,\lambda)$, (16) is rewritten as

$$U(\gamma,\lambda) = e^{\gamma}\int dq_1dq_2\left|\begin{pmatrix} q_1 \\ q_2\end{pmatrix}\right\rangle\left\langle R'\begin{pmatrix} q_1 \\ q_2\end{pmatrix}\right|, \quad R' = \begin{pmatrix} e^{\gamma}\cosh\lambda & e^{\gamma}\sinh\lambda \\ e^{\gamma}\sinh\lambda & e^{\gamma}\cosh\lambda \end{pmatrix}. \tag{18}$$

Putting (15) into (16) and using the IWOP technique [29, 30] as well as the normal ordering of vacuum projector

$$|0,0\rangle\langle 0,0| =: \exp\left[-a_1^\dagger a_1 - a_2^\dagger a_2\right]: , \tag{19}$$

where $: \ :$ denotes normal ordering, one can perform the integration and finally get the normally ordered expansion of $U(\gamma,\lambda)$, i.e.,

$$U(\gamma,\lambda) = \frac{1}{\sqrt{\kappa}}\exp\left\{-\frac{1}{4\kappa}\left[\sinh 2\gamma\left(a_1^{\dagger 2} + a_2^{\dagger 2}\right) + 2\sinh 2\lambda a_1^\dagger a_2^\dagger\right]\right\} \\ \times : \exp\left\{\frac{1}{\kappa}\left[(\cosh\lambda\cosh\gamma - \kappa)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) - \sinh\lambda\sinh\gamma\left(a_1^\dagger a_2 + a_2^\dagger a_1\right)\right]\right\} : \\ \times \exp\left\{\frac{1}{4\kappa}\left[\sinh 2\gamma\left(a_1^2 + a_2^2\right) + 2\sinh 2\lambda a_1 a_2\right]\right\}, \tag{20}$$

where

$$\kappa \equiv \cosh(\lambda+\gamma)\cosh(\lambda-\gamma). \tag{21}$$

Based on the normally ordered form of $U(\gamma,\lambda)$, we consider Yuen's way of generating squeezed states. Applying the squeezing operator $U(\gamma,\lambda)$ on the two-mode coherent state

$$|\alpha_1,\alpha_2\rangle = D(\alpha_1,\alpha_2)|0,0\rangle, \tag{22}$$

where $D(\alpha_1,\alpha_2)$ is the displacement operator with α_j being complex displacement parameter,

$$D(\alpha_1,\alpha_2) = \exp(\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger - \alpha_1^* a_1 - \alpha_2^* a_2), \tag{23}$$

one has the generalized two-mode squeezed state

$$|\phi\rangle = U(\gamma,\lambda)D(\alpha_1,\alpha_2)|0,0\rangle$$

$$= \frac{1}{\sqrt{\kappa}} \exp\left\{-\frac{1}{2}\left(|\alpha_1|^2+|\alpha_2|^2\right) - \frac{1}{4\kappa}\sinh 2\gamma\left(a_1^{\dagger 2}+a_2^{\dagger 2}-\alpha_1^2-\alpha_2^2\right)\right.$$
$$-\frac{1}{2\kappa}\sinh 2\lambda\left(a_1^{\dagger}a_2^{\dagger}-\alpha_1\alpha_2\right)+\frac{1}{\kappa}\cosh\lambda\cosh\gamma\left(a_1^{\dagger}\alpha_1+a_2^{\dagger}\alpha_2\right)$$
$$\left.-\frac{1}{\kappa}\sinh\lambda\sinh\gamma\left(a_1^{\dagger}\alpha_2+a_2^{\dagger}\alpha_1\right)\right\}|0,0\rangle. \tag{24}$$

When $\alpha_1=\alpha_2=0$, (24) recovers to the generalized two-mode squeezed vacuum state

$$U(\gamma,\lambda)|0,0\rangle = \frac{1}{\sqrt{\kappa}}\exp\left\{-\frac{1}{4\kappa}\left[\sinh 2\gamma\left(a_1^{\dagger 2}+a_2^{\dagger 2}\right)+2\sinh 2\lambda a_1^{\dagger}a_2^{\dagger}\right]\right\}|0,0\rangle \equiv |\phi\rangle_0. \tag{25}$$

3 Higher-Order Squeezing for $|\phi\rangle$

In this section, we turn our attention to evaluate the higher-order squeezing behavior for the generalized two-mode squeezed state $|\phi\rangle$. The concept of higher-order squeezing [32, 33], proposed by Hong and Mandel, is used to reveal nonclassical behavior of the higher-order moments in optical states, namely, when $2N$-th moment in a state is less than that in the coherent state, this state is said to be squeezed to order $2N$.

We first derive the expression of the $2N$-th moment of quadrature operators in the state $|\phi\rangle$. Following Loudon and Knight [34], we introduce the two-mode quadrature operators

$$X=\frac{1}{2}(Q_1+Q_2)=\frac{1}{2\sqrt{2}}\left(a_1+a_1^{\dagger}+a_2+a_2^{\dagger}\right), \tag{26}$$

and

$$Y=\frac{1}{2}(P_1+P_2)=\frac{1}{i2\sqrt{2}}\left(a_1-a_1^{\dagger}+a_2-a_2^{\dagger}\right), \tag{27}$$

with $[X,Y]=i/2$. From (6) and (7), we obtain

$$U^{-1}XU=\frac{1}{2\sqrt{2}}f\left(a_1+a_1^{\dagger}+a_2+a_2^{\dagger}\right) \tag{28}$$

and

$$U^{-1}YU=\frac{1}{2\sqrt{2}}g\left(a_1-a_1^{\dagger}+a_2-a_2^{\dagger}\right), \tag{29}$$

respectively, where $f=e^{-\gamma}(\cosh\lambda-\sinh\lambda)$ and $g=e^{\gamma}(\cosh\lambda+\sinh\lambda)$.

Letting $F=\mu_1a_1+\mu_2a_1^{\dagger}+\nu_1a_2+\nu_2a_2^{\dagger}$ and using $[a_j,a_k^{\dagger}]=\delta_{jk}$, ones can easily have

$$e^{\xi\Delta F}=:e^{\xi\Delta F}:e^{\frac{1}{2}\xi^2(\mu_1\mu_2+\nu_1\nu_2)} \tag{30}$$

where $\Delta F\equiv F-\bar{F}$, μ_j, ν_j and ξ are C number. By expanding both sides of (30) as power series in ξ, and equating coefficients of $(2N)!\xi^{2N}$, it is obtained that

$$(\Delta F)^{2N}=\sum_{k=0}^{N}\frac{(2N)!}{k!(2N-2k)!}\left(\frac{\mu_1\mu_2+\nu_1\nu_2}{2}\right)^k:(\Delta F)^{2N-2k}: \tag{31}$$

which is a basic formula in calculating $2N$-th moment of two-mode fields.

As a result of (28), (29) and (31), we have

$$
\begin{aligned}
&\langle\phi|(\Delta X)^{2N}|\phi\rangle \\
&=\left(\frac{1}{8}\right)^{N}\langle\alpha_1,\alpha_2|\left[\Delta\left(a_1+a_1^{\dagger}+a_2+a_2^{\dagger}\right)f\right]^{2N}|\alpha_1,\alpha_2\rangle \\
&=\left(\frac{1}{8}\right)^{N}\sum_{k=0}^{N}\frac{(2N)!f^{2k}}{k!(2N-2k)!}\langle\alpha_1,\alpha_2|:\left[\Delta\left(a_1+a_1^{\dagger}+a_2+a_2^{\dagger}\right)f\right]^{2N-2k}:|\alpha_1,\alpha_2\rangle \\
&=\left(\frac{1}{4}\right)^{N}(2N-1)!!e^{-2N(\gamma+\lambda)},
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
&\langle\phi|(\Delta Y)^{2N}|\phi\rangle \\
&=\left(-\frac{1}{8}\right)^{N}\langle\alpha_1,\alpha_2|\left[\Delta\left(a_1-a_1^{\dagger}+a_2-a_2^{\dagger}\right)g\right]^{2N}|\alpha_1,\alpha_2\rangle \\
&=\left(-\frac{1}{8}\right)^{N}\sum_{k=0}^{N}\frac{(2N)!(-g)^{2k}}{k!(2N-2k)!}\langle\alpha_1,\alpha_2|:\left[\Delta\left(a_1-a_1^{\dagger}+a_2-a_2^{\dagger}\right)g\right]^{2N-2k}:|\alpha_1,\alpha_2\rangle \\
&=\left(\frac{1}{4}\right)^{N}(2N-1)!!e^{2N(\gamma+\lambda)}.
\end{aligned}
\tag{33}
$$

So for the two-mode coherent state, namely, $\gamma=\lambda=0$,

$$
\langle\alpha_1,\alpha_2|\,(\Delta X)^{2N}\,|\alpha_1,\alpha_2\rangle=\langle\alpha_1,\alpha_2|\,(\Delta Y)^{2N}|\alpha_1,\alpha_2\rangle=\left(\frac{1}{4}\right)^{N}(2N-1)!!. \tag{34}
$$

According to the definition of higher-order squeezing, the generalized two-mode squeezed state $|\phi\rangle$ can be squeezed to all even orders for $\gamma+\lambda>0$. When $\gamma=0$, ones have

$$
\langle\phi|\,(\Delta X)^{2N}\,|\phi\rangle\big|_{\gamma=0}=\left(\frac{1}{4}\right)^{N}(2N-1)!!e^{-2N\lambda}, \tag{35}
$$

which is just the $2N$th moment for the usual two-mode squeezed state. Comparing (32) with (35), the state $|\phi\rangle$ can exhibit a stronger form of higher-order squeezing than the usual two-mode squeezed state.

In particular, when $N=1$, (32) and (33) reduce to the quadrature squeezing for the state $|\phi\rangle$, i.e.,

$$
\langle\phi|\,(\Delta X)^2|\phi\rangle=\frac{1}{4}e^{-2(\gamma+\lambda)},\qquad \langle\phi|\,(\Delta Y)^2|\phi\rangle=\frac{1}{4}e^{2(\gamma+\lambda)} \tag{36}
$$

and

$$
\langle\phi|\,(\Delta X)^2\,|\phi\rangle\,\langle\phi|\,(\Delta Y)^2|\phi\rangle=\frac{1}{16}. \tag{37}
$$

Obviously, the above relations indicate that $|\phi\rangle$ may also exhibit stronger squeezing $e^{-2(\gamma+\lambda)}$ in one quadrature than that $e^{-2\lambda}$ of the usual two-mode squeezed state. Therefore we call $U(\gamma,\lambda)$ the two-mode enhancing squeezing operator.

4 Entanglement Analysis for $|\phi\rangle$

A striking quantum phenomenon is the entanglement (inseparability) of composite quantum systems, whose most famous example is the violation of Bell's inequality. In this section, we analyze the entanglement for the state $|\phi\rangle$ using the total variance of a pair of EPR-like operators introduced by Duan group [35].

In [35], it has been proved that if a state is separable, the total variance of EPR-like operators u and v satisfy the inequality

$$M \equiv \langle(\Delta u)^2\rangle + \langle(\Delta v)^2\rangle \geq m^2 + \frac{1}{m^2}, \tag{38}$$

where

$$u = |m|Q_1 + \frac{Q_2}{m}, \tag{39}$$

and

$$v = |m|P_1 - \frac{P_2}{m}. \tag{40}$$

Clearly, it is a sufficient condition for entangled states when the inequality in (38) is violated. In addition, we always hope that the variances of EPR-like operators are as small as possible. Thus one can get the condition $m = 1$. In this case, the mean photon numbers of two modes are equal and the common eigenstate of EPR-type operator $Q_1 + Q_2$ and $P_1 - P_2$ is a maximum entangled states [36]. Then (38) may be expressed as

$$M \equiv \langle(\Delta u)^2\rangle + \langle(\Delta v)^2\rangle \geq 2, \tag{41}$$

which implies that it is a necessary and sufficient condition for entangled states when the inequality in (41) is violated.

For the generalized two-mode squeezed state $|\phi\rangle$ in (24), using (9)–(12), ones have

$$\begin{aligned}\langle u\rangle_{m=1} &= \langle\alpha_1,\alpha_2|U^{-1}(\gamma,\lambda)(Q_1+Q_2)U(\gamma,\lambda)|\alpha_1,\alpha_2\rangle \\ &= \frac{e^{-(\gamma+\lambda)}}{\sqrt{2}}\left(\alpha_1+\alpha_1^*+\alpha_2+\alpha_2^*\right),\end{aligned} \tag{42}$$

$$\langle u^2\rangle_{m=1} = \frac{e^{-2(\gamma+\lambda)}}{2}\left[\left(\alpha_1+\alpha_1^*\right)^2 + 2\left(\alpha_1+\alpha_1^*\right)\left(\alpha_2+\alpha_2^*\right) + (\alpha_2+\alpha_2)^2 + 2\right], \tag{43}$$

and

$$\left\langle(\Delta u)^2\right\rangle_{m=1} = e^{-2(\gamma+\lambda)}. \tag{44}$$

Similarly,

$$\left\langle(\Delta v)^2\right\rangle_{m=1} = e^{-2(\gamma-\lambda)}. \tag{45}$$

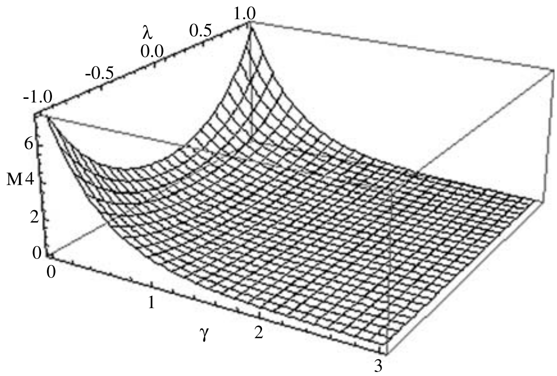

Fig. 1 The total variance $\langle(\Delta u)^2\rangle + \langle(\Delta v)^2\rangle$ with $m = 1$ as a function of γ and λ

It follows from (41) that

$$M = 2e^{-2\gamma}\cosh 2\lambda. \tag{46}$$

In order to clearly analyze the entanglement for the state $|\phi\rangle$, we plot the variance $\langle(\Delta u)^2\rangle + \langle(\Delta v)^2\rangle$ in (46) as a function of γ and λ in Fig. 1. We can observe $M < 2$ with larger γ for the smaller values of $|\lambda|$. Then the inequality in (38) are violated. And the bigger γ is, the larger violation is. Thus, we conclude that the generalized two-mode squeezed states are inseparable, namely, entangled states.

5 Wigner Function and Marginal Distribution for $|\phi\rangle$

It is well known that the Wigner function is a powerful tool to investigate the nonclassicality of optical fields. Its partial negativity implies the highly nonclassical properties of quantum state and is often used to describe the decoherence of quantum state [37, 38]. In this section, we want to derive the analytical expression of the Wigner function for the generalized two-mode squeezed state $|\phi\rangle$ and calculate its marginal distribution of the Wigner function.

Here, we use the Weyl ordering invariance under similar transformation to derive the Wigner function for $|\phi\rangle$. For this purpose, we first recall that the Weyl ordering form of single-mode Wigner operator $\Delta_1^w(q_1, p_1)$ is given by [39, 40]

$$\Delta_1^w(q_1, p_1) = \vdots\, \delta(q_1 - Q_1)\delta(p_1 - P_1) \,\vdots \tag{47}$$

and its normal ordering form is

$$\Delta_1^w(q_1, p_1) = \frac{1}{\pi} :\, \exp\left[-(q_1 - Q_1)^2 - (p_1 - P_1)^2\right] :, \tag{48}$$

where the symbols $: \, :$ and $\vdots \, \vdots$ denote the normal ordering and the Weyl ordering, respectively, in which the Boson operators a_1 and $a_1^\dagger$ can be permuted. q_1 and p_1 are the eigenvalues of the operators Q_1, P_1, respectively. For the two-mode case, the Weyl ordering form of the Wigner operator is

$$\Delta_2^w(\vec{q}, \vec{p}) = \vdots\, \delta(\vec{q} - \vec{Q})\delta(\vec{p} - \vec{P}) \,\vdots,$$

where $\vec{y} \equiv \binom{y_1}{y_2}$.

Then according to the Weyl ordering invariance under similar transformations proved in [41] and using (9)–(12), we may obtain the following result

$$\begin{aligned} U^{-1}\Delta_2^w(\vec{q},\vec{p})\,U &= U^{-1}\,\vdots\,\delta(\vec{q}-\vec{Q})\delta(\vec{p}-\vec{P})\,\vdots\,U \\ &= \vdots\,\delta\left[q_k - R_{kj}Q_j\right]\delta\left[p_k - \left(R^{-1}\right)_{kj}P_j\right]\vdots \\ &= \vdots\,\delta\left[\left(R^{-1}\right)_{jk}q_k - Q_j\right]\delta\left[R_{jk}p_k - P_j\right]\vdots \\ &= \Delta_2^w\left(R^{-1}\vec{q}, R\vec{p}\right), \end{aligned} \tag{49}$$

where $k, j \in \{1, 2\}$ and from (17), R^{-1} equals to

$$R^{-1} = \begin{pmatrix} e^{\gamma}\cosh\lambda & e^{\gamma}\sinh\lambda \\ e^{\gamma}\sinh\lambda & e^{\gamma}\cosh\lambda \end{pmatrix}. \tag{50}$$

Note that we have used the Einstein summation notation. Therefore, by considering (47) and (48) and using (49), the Wigner function of $|\phi\rangle$ is calculated as

$$\begin{aligned} W(z_1,z_2) &\equiv W(q_1,p_1,q_2,p_2) \\ &= \frac{1}{\pi^2}\langle\alpha_1,\alpha_2|\,U^{-1}(\gamma,\lambda)\,\Delta_2^w(\vec{q},\vec{p})\,U(\gamma,\lambda)\,|\alpha_1,\alpha_2\rangle \\ &= \frac{1}{\pi^2}\langle\alpha_1,\alpha_2|:\ \exp\left[-\left(R^{-1}\vec{q}-\vec{Q}\right)^2-\left(R\vec{p}-\vec{P}\right)^2\right]:\ |\alpha_1,\alpha_2\rangle \\ &= \frac{1}{\pi^2}e^{-2|\alpha_1|^2-2|\alpha_2|^2}\exp\left[-e^{2\gamma}\cosh 2\lambda\left(q_1^2+q_2^2\right)-e^{-2\gamma}\cosh 2\lambda\left(p_1^2+p_2^2\right)\right. \\ &\quad -2e^{2\gamma}\sinh 2\lambda q_2q_1+2e^{-2\gamma}\sinh 2\lambda p_2p_1+\sqrt{2}\left(M_1q_1+M_2q_2\right) \\ &\quad \left.+i\sqrt{2}\left(N_2p_2+N_1p_1\right)\right], \end{aligned} \tag{51}$$

where $q_j = \frac{1}{\sqrt{2}}(z_j + z_j^*)$ and $p_j = \frac{1}{i\sqrt{2}}(z_j - z_j^*)$, M_1, M_2, N_1 and N_2 denote the following equations, respectively,

$$M_1 \equiv e^{\gamma}\cosh\lambda\left(\alpha_1+\alpha_1^*\right)+e^{\gamma}\sinh\lambda\left(\alpha_2+\alpha_2^*\right), \tag{52}$$

$$M_2 \equiv e^{\gamma}\sinh\lambda\left(\alpha_1+\alpha_1^*\right)+e^{\gamma}\cosh\lambda\left(\alpha_2+\alpha_2^*\right), \tag{53}$$

$$N_1 \equiv e^{-\gamma}\sinh\lambda\left(\alpha_2-\alpha_2^*\right)-e^{-\gamma}\cosh\lambda\left(\alpha_1-\alpha_1^*\right), \tag{54}$$

$$N_2 \equiv e^{-\gamma}\sinh\lambda\left(\alpha_1-\alpha_1^*\right)-e^{-\gamma}\cosh\lambda\left(\alpha_2-\alpha_2^*\right). \tag{55}$$

Next, based on the above results, we calculate the marginal distribution of Wigner operator in the state $|\phi\rangle$. The marginal distribution of Wigner function in the q-direction is derived as

$$\begin{aligned} W_q &= \iint dp_1dp_2W(q_1,p_1,q_2,p_2) \\ &= \frac{1}{\pi^2}e^{-2|\alpha_1|^2-2|\alpha_2|^2}\exp\left[-e^{2\gamma}\cosh 2\lambda\left(q_1^2+q_2^2\right)-2e^{2\gamma}\sinh 2\lambda q_2q_1\right. \end{aligned}$$

$$
\begin{aligned}
&+\sqrt{2}\left(M_1q_1+M_2q_2\right)\Big]\\
&\times\iint dp_1dp_2\exp\Big[-e^{-2\gamma}\cosh 2\lambda p_1^2+i\sqrt{2}N_1p_1+2e^{-2\gamma}\sinh 2\lambda p_2p_1\\
&-e^{-2\gamma}\cosh 2\lambda p_2^2+i\sqrt{2}N_2p_2\Big]\\
=&\frac{e^{2\gamma}}{\pi}\exp\left[-2\left|\alpha_1\right|^2-2\left|\alpha_2\right|^2-\frac{1}{2}\left(N_1^2+N_2^2\right)e^{2\gamma}\cosh 2\lambda-N_1N_2e^{2\gamma}\sinh 2\lambda\right]\\
&\times\exp\left[-e^{2\gamma}\cosh 2\lambda\left(q_1^2+q_2^2\right)-2e^{2\gamma}\sinh 2\lambda q_2q_1+\sqrt{2}\left(M_1q_1+M_2q_2\right)\right]. \qquad (56)
\end{aligned}
$$

Similarly, its marginal distribution in the p-direction is

$$
\begin{aligned}
W_p=&\iint dq_1dq_2W(q_1,p_1,q_2,p_2)\\
=&\frac{1}{\pi^2}e^{-2|\alpha_1|^2-2|\alpha_2|^2}\exp\Big[-e^{-2\gamma}\cosh 2\lambda\left(p_1^2+p_2^2\right)+2e^{-2\gamma}\sinh 2\lambda p_2p_1\\
&+i\sqrt{2}\left(N_2p_2+N_1p_1\right)\Big]\\
&\times\iint dq_1dq_2\exp\Big[-e^{2\gamma}\cosh 2\lambda q_1^2-2e^{2\gamma}\sinh 2\lambda q_2q_1+\sqrt{2}M_1q_1\\
&-e^{2\gamma}\cosh 2\lambda q_2^2+\sqrt{2}M_2q_2\Big]\\
=&\frac{e^{-2\gamma}}{\pi}\exp\left[-2\left|\alpha_1\right|^2-2\left|\alpha_2\right|^2+\frac{1}{2}\left(M_2^2+M_1^2\right)e^{-2\gamma}\cosh 2\lambda-M_1M_2e^{-2\gamma}\sinh 2\lambda\right]\\
&\times\exp\left[-e^{-2\gamma}\cosh 2\lambda\left(p_1^2+p_2^2\right)+2e^{-2\gamma}\sinh 2\lambda p_2p_1+i\sqrt{2}\left(N_2p_2+N_1p_1\right)\right]. \qquad (57)
\end{aligned}
$$

Here we have used the following integral formula

$$
\int dx\exp\left(-ax^2+bx\right)=\left(\frac{\pi}{a}\right)^{1/2}\exp\left(\frac{b^2}{4a}\right),\quad (a>0). \qquad (58)
$$

Especially, when $\alpha_1=\alpha_2=0$, (51) becomes

$$
\begin{aligned}
&W(q_1,p_1,q_2,p_2)|_{\alpha_1=\alpha_2=0}\\
&=\frac{1}{\pi^2}\exp\left[-e^{2\gamma}\cosh 2\lambda\left(q_1^2+q_2^2+p_1^2+p_2^2\right)-2e^{2\gamma}\sinh 2\lambda\left(q_2q_1-p_2p_1\right)\right], \qquad (59)
\end{aligned}
$$

which is the Wigner function of the generalized two-mode squeezed vacuum state. On the other hand, by letting $\alpha_1=\alpha_2=0$ and $\gamma=0$, it is obtained from (51) that the Wigner function of the usual two-mode squeezed vacuum state is

$$
W=\frac{1}{\pi^2}\exp\left[-\left(q_1^2+q_2^2+p_1^2+p_2^2\right)\cosh 2\lambda-2\left(q_2q_1-p_2p_1\right)\sinh 2\lambda\right], \qquad (60)
$$

which manifestly exhibits the entanglement between (q_1,p_1) and (q_2,p_2).

6 Conclusions

In summary, we have presented a class of the generalized two-mode squeezed states $|\phi\rangle$, generated from the generalized two-mode squeezing operator $U(\gamma,\lambda)$ acting on the two-mode coherent state $|\alpha_1,\alpha_2\rangle$. We investigate some mathematical properties of $U(\gamma,\lambda)$ including the squeezing transformation under $U(\gamma,\lambda)$, ket-bra integral form in the coordinate representation, normally ordered form. Then we evaluate some nonclassical characteristics of the state $|\phi\rangle$ such as higher-order squeezing behavior and entanglement analysis. It is found that the state $|\phi\rangle$ can exhibit a stronger form of higher-order squeezing than the usual two-mode squeezed state and the state $|\phi\rangle$ are inseparable with larger γ for the smaller values of $|\lambda|$. Finally, we derive the analytical expression of the Wigner function for the state $|\phi\rangle$ by using the Weyl ordering invariance under similar transformation. This approach seems very simply.

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